Shape Analysis from Complex Systems using Information Geometry tools

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In this paper we model statistically the patterns from Complex Systems and describe their evolution in time, using Information Geometry tools. In particular we propose an index capable to estimate the level of self-organization and foresee future problems in the systems.

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I. INTRODUCTION

One of the most important properties of Complex Systems is the forming of patterns: a band of fish in the sea, a flight of birds in the sky, the dunes in the desert are only few examples, [2, 3]. These patterns are not shapes fixed in time but they evolve with the dynamics of the Complex Systems and vary according to their level of self-organization. The aim of this paper is modeling statistically the patterns from Complex Systems and describing step by step their change, using Information Geometry tools. In particular we wish to discover an index which is capable to understand the trend in the self-organization phenomenon and capture eventual signals of crisis.

As example of a complex system, we can consider the macula, that is the central part of the retina in the eye, where the distinct vision occurs. A frequent pathology, due to the degrading of that complex system, is the irreversible reduction of the vision in the people more of 65 years old, which is called macular degeneration due to the age. Ophthalmologists distinguish the evolution of the disease in two phases: an initial form, called dry, and a terminal form, which can be new-vascular or for atrophy. The first is characterized by the presence of some retina'a damages, such as the drusen which is an accumulation of lipoproteins, and areas of change of the pigmentation of the epithelium. In this phase people continues to have a discreet level of vision. On the contrary, the second phase produces a serious loss of vision's capacity and it is characterized by the appearance of a central scotoma produced by the development of anomalous new-vessels near the macula. These blood-vessels arise in the core and go to the retina and sub-retina's stratum, producing a vascular scar responsible of destruction of retina's center. The atrophy is characterized by loss of the normal retina's stratus. The diagnosis of macular degeneration is made observing the ocular fundus by ophthalmoscopy and using recent imaging techniques, such as fluor-angiography, angiography with green of indiocian and optical coherence tomography stratus domain. Every of such techniques consents the view of the typical damages, their classification, supervision in time and this is very useful to value the efficacy of the therapies. Using the fluor-angiography technique, at the left side of the figure we see a normal shape of this Complex System.

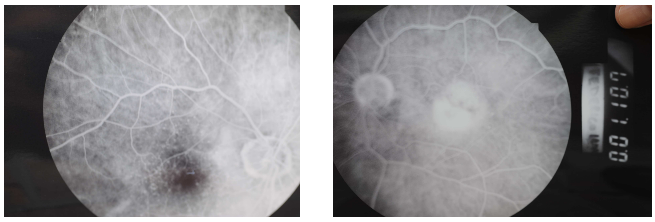

FIG. 1: Left side: Normal case. Right side: Terminal case.

In the first phase, usually this shape does not change, but we can observe blots in the image. At the right side of the figure we observe a typical second phase, where new-forms join the initial pattern.

II. STATISTICAL MANIFOLDS AND MODELS OF COMPLEX SHAPES

It is well-known that the object of study in the Information Geometry, [4, 5], are the Statistical Manifolds, which are families of probability density

$$\Theta = \{p(x/\theta) : x = (x_1, ..., x_m), \theta = (\theta_1, ..., \theta_m)\} \quad (1)$$

with its local coordinates defined by the model parameters θ. For example, a univariate Gaussian density can be represented as a single point on 2-dimensional manifold with coordinates $\theta = (\mu, \sigma)$, $\mu \in \mathbb{R}$ and $\sigma > 0$ (superior half-plane), where as usual these represent the mean and the standard deviation of the density. The most famous Statistical Manifolds are the Exponential Families, among those there are the Normal and Poisson Families and also the Mixture Family.

Data from a complex pattern are often realized as a set of points. Many statistical methods, such as *manual assignment* or *clustering*, allow us to extract some points which are representative for the shape and are called *landmarks*. We can mean these landmarks as the centers of the surrounding points. From this point of view,

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Peter and Rangarajan, [1], identify k landmarks of the same shape with the mean points of a k-component Gaussian mixture model (GMM). In particular, if we consider planar shapes, the landmarks positions are the means for the specific bi-variate components of the (GMM). Therefore they write

$$p(x/\theta) = \frac{1}{2\pi\sigma^2 K}\sum_{i=1}^{K}\exp\{-\frac{\|x-\mu^i\|^2}{2\sigma^2}\} \quad (2)$$

where θ is the set of all the landmarks $\mu^i = (\mu_1^i, \mu_2^i)$ and $x = (x_1, x_2)$.

In the absence of any a priori-knowledge, it is acceptable to put in the model equal weight $\frac{1}{K}$ to every landmark. The variance σ captures uncertainties that arise in placement of landmarks and/or the natural variability across a population of shapes. It provides the flexibility to model complex patterns. So Peter and Rangarajan consider σ as a free parameter, which is isotropic across all components. Therefore they only use the means of a (GMM) as the coordinates of the manifold.

In this paper, on the contrary, variances are considered as further coordinates for the landmarks of a complex shape, compatibly with the Information Geometry theory. Is is clear that a landmark with a big variance informs us that it is not much representative of its surrounding points. Besides we also remove the isotropic hypothesis. Indeed, for example, when we have a photograph unfocused on its part, we can not state that the information message we deduce is uniform. The model of Peter and Rangarajan, even if numerically more simple, induces a loss of information in the Fisher sense. But, in some cases, it is reasonable to use it, in particular when the change of the shape is due to external forces. Indeed the Authors refer to "deformation of the external space" and unify representation and deformation. On the contrary, we are interested in the natural evolution of the shape produced by "internal forces" to the system. This is very important, for example in medicine, indeed often the dimmed and stained imagine is the warning of some problem for the involved organ. Therefore we consider the following model:

$$p(x/\theta) = \frac{1}{2\pi K}\sum_{i=1}^{K}\frac{1}{\|\sigma^i\|^2}\exp\{-\frac{\|x-\mu^i\|^2}{2\|\sigma^i\|^2}\} \quad (3)$$

where $\theta = \{\theta^i = (\mu^i, \sigma^i) : i = 1, ..., K\}$ with $\sigma^i = (\sigma_1^i, \sigma_2^i)$ and the other symbols are as in the previous case.

III. FISHER-RAO METRIC AND EVOLUTION OF COMPLEX SHAPES

From Information Geometry we also know that Fisher information matrix induces a riemannian metric on the Statistical Manifold, called the Fisher-Rao metric, with metric tensor:

$$g_{ij}(\theta) = \int p(x/\theta)\frac{\partial}{\partial\theta^i}\log p(x/\theta)\frac{\partial}{\partial\theta^j}\log p(x/\theta)dx \quad (4)$$

In the univariate gaussian case it is possible to prove that the Fisher-Rao metric induces the hyperbolic geometry.

Differential Geometry proves that all the analysis is intrinsic, that is on the manifold without considering how it embeds in an euclidean space. Besides we can search for a curve for two points or a point with a tangent vector, which minimizing locally the distance induced by the metric. These paths, called geodesics, are obtainable as solutions of the following nonlinear system of partial differential equations

$$g_{ki}\ddot{\theta}^i + \Gamma_{k,ij}\dot{\theta}^i\dot{\theta}^j = 0 \quad (5)$$

where

$$\Gamma_{k,ij} = \frac{1}{2}\{\frac{\partial g_{ik}}{\partial\theta^j} + \frac{\partial g_{kj}}{\partial\theta^i} - \frac{\partial g_{ij}}{\partial\theta^k}\} \quad (6)$$

is the Christoffel symbol of the first kind.

Then, given two points on the Statistical Manifold, there exists only one geodesic with connects them minimizing the Fisher information. We deduce that, if two shapes are represented by mixture models, the parameters of which map points on the Statistical Manifold, it is possible to use Fisher-Rao metric to construct a geodesic between them which will inform us on the intermediate shapes (landmarks and their variances), that is it will allow us to reconstruct the steps from one shape onto another. We note that, in the (GMM) case, the geodesic equations induce a system not analytically solvable, so we have to use numerical techniques. That intrinsic path will drive the reconstruction of the real points which constitute the intermediate shapes in the external space. Precisely, the intrinsic evolution of the landmarks and their variances requires the parametrization of θ by time:

$$p(x/\theta(t)) = \frac{1}{2\pi K}\sum_{i=1}^{K}\frac{1}{\|\sigma^i(t)\|^2}\exp\{-\frac{\|x-\mu^i(t)\|^2}{2\|\sigma^i(t)\|^2}\} \quad (7)$$

We note that $\sigma^i(t)$ analysis gives us information regarding the dispersion of the real points of the shape around their means $\mu^i(t)$, when t is varying. If $\sigma^i(t)$ increase in time, we lose detailed resemblance to the original shape and, when it is a complex pattern, we can deduce a degrading of the self-organization as connecting phenomenon of the system. Numerical simulations prove that, in this case, the image shows blots, as a photocopy from a damaged machine. Besides, the study of the instantaneous speed of $\theta^i(t)$, allows us a forecast, in the short time, of the evolution of the pattern and of the eventual tendency to break up of the system.

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